General Pseudoadditivity of Kapur's Entropy prescribed by the existence of equilibrium

Priti Gupta, Vijay Kumar

Abstract—In this paper, the concept of composibility of the total channel capacity of the system of channels composed of two independent subsystems of channel is discussed. This system of channels is a function of entropies of the subsystems. Here, we show that the generalized entropies, the Tsalli's entropy, the normalized Tsalli's entropy and the Kapur's entropy are compatible with the composition law and defined consistently with the condition of equilibrium and satisfies pseudo additivity.

Key words— Entropy, Kapur's Entropy, Pseudo-additivity

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1 INTRODUCTION

In the process of transmission of signals, a communication system has to do with the amount of information, the capacity of the communication channel and the effects of noise. Communication is a continuous process and a channel is the 'pipe' along which a information is conveyed, and there are a wide variety of different communication channels available, from basic face-to-face conversation, through to telecommunication channels like the telephone or e-mail and computational channels like the medical record. Channels have attributes like capacity and noise, which determine their suitability for different tasks. When two parties exchange information across a channel at the same time, this is known as *synchronous communication.*

For example:: Telephones are one of the commonest two-way synchronous channels. It is the nature of synchronous communication that it is interruptive, and these interruptions may have a negative impact on individuals who have high cognitive loads.

On the contrary, when individuals can be separated in time, they may use an *asynchronous* channel to support their interaction. Since there can be no simultaneous discussion, conversations occur through a series of message exchanges. This can range from Post-it notes left on a colleague's desk, to sophisticated electronic information systems. One of the benefits of asynchronous communication is that it is not inherently interruptive, and if a communication is not urgent, asynchronous channels may be a preferred way of communicating with otherwise busy individuals.

A communication system is thus a bundle of different components and the utility of the overall system is

determined by the appropriateness of all the components together.

If even one element of the system bundle is inappropriate to the setting, the communication system can under perform. A communication system provides only a channel for mutual information exchange which is not apriori dedicated to certain categories of information only.

The channels can be modeled physically by trying to calculate the physical processes which modify the transmission of signals.

For example:: In wireless communications the channel is often modeled by a random attenuation (known as fading) of the transmitted signal, followed by additive noise. It can be modeled by calculating the reflection of every object in the environment.

The attenuation term is a simplification of the underlying physical processes and captures the change in signal power over the course of the transmission. The noise in the model captures external interference and/or electronic noise in the receiver. If the attenuation term is complex, it describes the relative time that a signal takes to get through the channel. The statistics of the random attenuation are decided by previous measurements or physical simulations.

Channel models may be continuous channel models in that there is no limit to how precisely their values may be defined.

As of today, it is now possible to assign multiple custom channels to a single ad unit. This feature enables us to track our ad performance with greater flexibility and view more granular information. When generating our ad code, we will be able to add up to five custom channels to a specific instance of ad code. Multiple channels can be very useful

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- *Dr. Priti Gupta is a professor in the Department of Staistics, Maharishi Dayanand University, Rohtak.*

- *Mr. Vijay Kumar is Senior Lecturer in Mathematics, Delhi College of Technology & Management, Palwal*

when we want to track one ad unit across several different metrics simultaneously.

For example: To run a sports website and we have to place a leader board at the top and bottom of every page. To track the performance of the ad placement, we have to create two custom channels - 'Top Leader board' and 'Bottom Leader board' and regenerated our ad code appropriately.

Also, if we also want to compare your football pages and your baseball pages at the same time. With multiple custom channels, this is not a problem. Just create two new custom channels called 'Football Pages' and 'Baseball Pages' and add them to the appropriate ad units. Now, each leader boards will be tagged with two custom channels that let us know the position they are in (top or bottom), and the type of page on which they appear (football or baseball). Moreover, ad units tagged with multiple custom channels will log impressions or clicks in each channel. As a result, we will see large number of impressions and clicks when we view channel reports than aggregate reports.

Generally, you can hear the static hiss on a radio or see it as snow on a television screen. Shannon realized that the noise added to one digital pulse would generally make the overall amplitude different from that of another, otherwise identical, pulse. Further, the noise amplitudes for two pulses are independent.

In the present work, we are trying to discuss the additivity (noise) of subsystems of channels. Here, we are considering two subsystems of a channel.

The additive requirement for the system(channel) puts constraints on the symmetry of the system under consideration: it is indivisibly connected with the homogeneity of the system, which is a familiar additional assumption in ordinary circumstances . Channels exhibit complex systems/processes and attracting great attention. A common feature of these systems is that they stay in non equilibrium stationary states for significantly long periods. In these systems, channels are generally in homogeneous.

In this paper, we discuss a general composition law for the system and satisfies pseudoadditivity. There, three generalized entropies, the Tsalli's entropy, the normalized Tsalli's entropy, and the Kapur's entropy, are shown to be consistent with the equilibrium condition. Abe [9] discuss the thermodynamic system, but here we are taking the system of channels for the same entropies.

From a thermodynamic viewpoint, the heat disturbs the signal, making it noisy.

1.1 KAPUR'S ENTROPY

Another characterization, which obtains for a partition of probability space for two segments, one for low valued probabilities and other for high valued probabilities, of pseudo-additivity of sub-systems A and B is:

$$H_\alpha^{(K)}(p_1, p_2, ..., p_w) = H_\alpha^{(K)}(p_l, p_m) + (p_l)^\alpha H_\alpha^{(K)}\left(\frac{p_1}{p_l}, \frac{p_2}{p_l}, ..., \frac{p_w}{p_l}\right)$$

$$+ (p_m)^\alpha H_\alpha^{(K)}\left(\frac{p_1}{p_m}, \frac{p_2}{p_m}, ..., \frac{p_w}{p_m}\right)$$

Where $p_l = \sum_{i=1}^{w_l} p_i$, $p_m = \sum_{i=w_l+1}^{w} p_i$ with $w_l + w_m = w$

and the sets $\left\{\frac{p_i}{p_l}\right\}$ and $\left\{\frac{p_i}{p_m}\right\}$ are conditional probabilities.

Also, $p_i^\alpha > p_i$ iff $\alpha < 1$ and $p_i^\alpha < p_i$ iff $\alpha > 1$

Here, $H_\alpha^{(K)}(p_1, p_2, ..., p_w)$ is of the form of a homogeneous function.

2. COMPOSABILITY OF VARIOUS ENTROPIES PRESCRIBED BY THE EXISTENCE OF EQUILIBRIUM

Let $X(A, B)$ be the total channel capacity of the system of channels and is divided into two independent subsystems of channels, A and B.

Let $X(A, B)$ satisfy additivity as:
$$X(A, B) = X(A) + X(B)$$
The joint probability of the total system of channel is

$$P_{ij}(A, B) = P_i(A) P_j(B) \qquad (2.1)$$

Where $p_i(A)$ is the probability of finding the subsystem A in its ith state and

$p_j(B)$ is the probability of finding the subsystem B in its jth state.

The entropy of the total system, $H(A, B) \equiv H[p(A, B)]$,

is expressed in terms of the entropies of the subsystems, $H(A)$ and $H(B)$, i.e,

$$H(A, B) = f\big[H(A), H(B)\big]$$

Where f is symmetrical bivariate function:

$$f(x, y) = f(y, x) \tag{2..2}$$

The given function is a composite function.

According to Boltzmann-Gibbs-Shannon entropy [3]

$$H(p) = -\sum_{i=1}^{N} p_i \ln(p_i) \tag{2.3}$$

Where N is the total no. of states at a given scale.

For independent variables A and B,

$$H(A, B) = H(A) + H(B) \tag{2.4}$$

From [8],

$$H(A_\alpha, B_\alpha) = H_\alpha(A) + H_\alpha(B) + \tau(\alpha) H_\alpha(A).H_\alpha(B)$$

Where $\tau(\alpha)$ is a function of the entropic index.

According to Renyi's Entropy [1]

$$H_\alpha^{(R)}(p) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{N} (p_i)^\alpha \quad (\alpha > 0) \tag{2.5}$$

and Tsalli's Entropy [4]

$$H_\alpha^{(T)}(p) = \frac{1}{1-\alpha} \left[\sum_{i=1}^{N} (p_i)^\alpha - 1 \right] \quad (\alpha > 0) \tag{2.6}$$

Since $H_\alpha^{(R)}(p)$ and $H_\alpha^{(T)}(p)$ are composite functions and satisfies pseudoadditivity as:

$$H_\alpha^{(T)}(A, B) = H_\alpha^{(T)}(A) + H_\alpha^{(T)}(B)$$
$$+ (1-\alpha) H_\alpha^{(T)}(A) H_\alpha^{(T)}(B) \tag{2.7}$$

The normalized Tsalli's Entropy is

$$H_\alpha^{(NT)}(p) = \frac{H_\alpha^{(T)}(p)}{\sum_{i=1}^{N} (p_i)^\alpha}$$

$$= \frac{1}{1-\alpha}\left[1 - \frac{1}{\sum_{i=1}^{N}(p_i)^\alpha} \right] \tag{2.8}$$

Also, $H_\alpha^{(NT)}(A, B) = H_\alpha^{(NT)}(A) + H_\alpha^{(NT)}(B)$
$$+ (\alpha - 1)H_\alpha^{(NT)}(A)H_\alpha^{(NT)}(B) \tag{2.9}$$

gives pseudoadditivity as (2.7)

Here, we notice two important points:

(i)   As $\alpha \to 1$, Renyi's Entropy, Tsalli's Entropy and normalized Tsalli's Entropy all converges to Boltzmann-Gibbs-Shannon Entropy.

(ii)   For $\alpha \in (0,1)$, Renyi's Entropy and normalized Tsalli's Entropy are concave only and for $\alpha > 0$, Tsalli's Entropy is always concave.

The generalized entropies of Kapur of order α and type β [7] is

$$H_{\alpha,\beta}^{K}(p) = \frac{1}{\beta - \alpha} \log_2 \left(\frac{\sum_{i=1}^{N} p_i^{\;\alpha}}{\sum_{i=1}^{N} p_i^{\;\beta}} \right),$$

$$\alpha \neq \beta, \ \alpha, \beta > 0 \tag{2.10}$$

In the limiting case,

When $\alpha \to 1$ and $\beta = 1$, $H_{\alpha,\beta}^{K}(p)$ reduces to $H(p)$ and when $\beta = 1$, $H_{\alpha,\beta}^{K}(p)$ reduces to $H_\alpha^{(R)}(p)$.

Also, $H_{\alpha,\beta}^{K}(p)$ is a composite function which satisfies pseudoadditivity as:

$$H_{\alpha,\beta}^{K}(A, B) = H_{\alpha,\beta}^{K}(A) + H_{\alpha,\beta}^{K}(B)$$

$$+ \tau(\alpha, \beta) \, H^{K}_{\alpha,\beta}(A) \cdot H^{K}_{\alpha,\beta}(B) \qquad (2.10\ a)$$

The above Entropies have a common feature " Sum plus product " as:

$$f(x, y) = x + y + \tau(\alpha) \, xy$$

Where $\tau(\alpha)$ is the frequency of channel .

For Renyi's Entropy and Boltzmann-Gibbs Shannon Entropy , $\tau(\alpha) = 0$

For Tsalli's Entropy, $\tau(\alpha) = 1 - \alpha$

For Normalized Tsalli's Entropy , $\tau(\alpha) = \alpha - 1$ and

Assuming that, For Kapur's Entropy,

$$\tau(\alpha, \beta) = (1 - \alpha)(1 - \beta)$$

The additivity of channel capacity is not obvious, this makes the channel highly non-trivial for non-extensive systems with long range interactions.

The stationary state of the total system may be defined by the maximization of total entropy with fixed $X(A, B)$, $i.e$,

$$\delta \, H(A, B) = \delta \, f\big[H(A), H(B)\big] = 0 \qquad (2.11)$$

$$\delta \, X(A) = - \delta \, X(B) \qquad (2.12)$$

From these equations it follows that,

$$\frac{\partial f\big[H(A), H(B)\big]}{\partial(H(A))} \cdot \frac{d(H(A))}{d(X(A))}$$
$$= \frac{\partial f\big[H(A), H(B)\big]}{\partial(H(B))} \cdot \frac{d(H(B))}{d(X(B))} \qquad (2.13)$$

In equilibrium state, $\tau(A) = \tau(B)$ \qquad (2.14)

Where τ is the frequency, if X is the channel capacity of the system

Separation of variables can only be possible if the derivation of the composite function satisfy

$$\frac{\partial f\big[H(A), H(B)\big]}{\partial(H(A))}$$
$$= k(H(A), H(B)) \cdot g(H(A)) \cdot h(H(B)) \quad (2.15)$$

$$\frac{\partial f\big[H(A), H(B)\big]}{\partial(H(B))}$$
$$= k(H(A), H(B)) \cdot h(H(A)) \cdot g(H(B)) \quad (2.16)$$

where $k, g \ and \ h$ are some functions. Also k is a symmetrical function due to (2..2)

In order to establish (2.14), in most physical systems k is independent of the entropies of subsystems and the possibility of k to be entropy dependent is limited, in general.

Henceforth, k will be taken to be an unity.

Equations (2.15) and (2.16) becomes as:

$$\frac{\partial f\big[H(A), H(B)\big]}{\partial(H(A))} = g(H(A)) \cdot h(H(B)) \qquad (2.17)$$

$$\frac{\partial f\big[H(A), H(B)\big]}{\partial(H(B))} = h(H(A)) \cdot g(H(B)) \qquad (2.18)$$

It is obvious that

$$\frac{\partial^2 f\big[H(A), H(B)\big]}{\partial(H(A)) \cdot \partial(H(B))} = \frac{\partial^2 f\big[H(A), H(B)\big]}{\partial(H(B)) \cdot \partial(H(A))} \quad (2.19)$$

Differentiate (2.17) with $H(B)$ and (2.18) with $H(A)$ and put it in (2.19), we get

$$\frac{1}{g(H(A))} \cdot \frac{dh(H(A))}{d(H(A))} = \frac{1}{g(H(B))} \cdot \frac{dh(H(B))}{d(H(B))}$$

$$\equiv \tau \qquad (2.20)$$

Where τ is a frequency and is known as separation constant.

Put the value of $g(H(A))$ and $g(H(B))$ in (2.17) and (2.18), we get

$$\frac{\partial f_\tau\big[H(A),H(B)\big]}{\partial(H(A))}$$
$$= \frac{1}{\tau}\cdot\frac{dh_\tau(H(A))}{d(H(A))}\cdot h_\tau(H(B)) \qquad (2.21)$$

$$\frac{\partial f_\tau\big[H(A),H(B)\big]}{\partial(H(B))}$$
$$= \frac{1}{\tau}\cdot\frac{dh_\tau(H(B))}{d(H(B))}\cdot h_\tau(H(A)) \qquad (2.22)$$

Integrating (2.21) with $H(A)$ and (2.22) with $H(B)$, we get

$$f_\tau\big[H(A),H(B)\big]$$
$$= \frac{1}{\tau}\cdot h_\tau(H(A))\cdot h_\tau(H(B)) + c \qquad (2.23)$$

Where c is the constant of integration.

In order to get convergent result in the limit $\tau \to 0$,

Without loss of generality, Setting $c = -\dfrac{1}{\tau}$

and simultaneously impose the condition

$$lt \qquad h_\tau(H)=1, \quad \forall H$$
$$\tau \to 0 \qquad\qquad\qquad\qquad (2.24)$$

Thus, $f_\tau\big[H(A),H(B)\big]$

$$= \frac{1}{\tau}\cdot h_\tau(H(A))\cdot h_\tau(H(B)) - \frac{1}{\tau} \quad (2.25)$$

Eq. (2.25) is the form of the composite function prescribed by the existence of equilibrium.

In order to find h_τ,

let us suppose that both A and B are in the completely ordered state, $i.e,$

$$H(A,B) = 0,$$

$$\therefore \quad h_\tau(0)=1 \qquad \forall\, \tau$$

Again, let us suppose that only B is in the completely ordered state, $i.e,$

$$H(A,B) = H(A) \qquad (2.26)$$

By (2.25), we get

$$H(A) = \frac{1}{\tau}\cdot h_\tau(H(A)) - \frac{1}{\tau} \qquad (2.27)$$

Or $\qquad h_\tau(H)=\tau\,H + 1 \qquad\qquad (2.28)$

Put (2.26) in (2.23), we get

$$H(A,B) = f\big[H(A),H(B)\big]$$

$$= H(A) + H(B) + \tau\,H(A)\cdot H(B) \qquad (2.29)$$

Eq. (2.29) is similar to eq. (2.10a).

This shows the validity of composition law of the system for Kapur's entropy and satisfies pseudoadditivity.

3. CONCLUSION

Here, we have derived the most general pseudoadditivity rule for composite entropy based only on the existence of equilibrium. We have shown how the three generalized entropies type pseudoadditivity can be obtained as the simplest case. In the present work, composability of entropy has been taken as a basic premise. This concept puts a stringent constraint on possible forms of entropies. In [10] the entropy arising from the idea of quantum groups is not composable. Composability does not seem to be possible to define the equilibrium states in the standard manner, if entropy does not satisfy composability.
However, it is worth noting that composability assumes divisibility of the total system into independent subsystems. Actually reliability of this independence also puts stringent constraints on physical nature of the systems.

For example: If the total system contains a long range interactions between its microscopic components, the independence of the subsystems may hardly be realiable in general. In such a situation, thorough generalization of the standard formalism may be required.

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